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# First Hyper Zagreb Spectral Radii of Splitting and Shadow Graphs

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**Abstract**

The spectral radius $\mathcal{R}_S$ of graph $G$ is a spectral invariant derived from the eigenvalues of the associated matrix for a graph $G$. It is widely used in fields such as computer science, chemistry, biology, and network analysis. A notable expansion of this notion is the First Hyper Zagreb spectral radius, $\mathcal{R}_S HZ_1(G)$, which is defined as the largest absolute eigenvalues of the First Hyper Zagreb matrix. This study examines the behaviour of First Hyper Zagreb spectral radius under two graph operations, generalized shadow and generalized splitting. We compare $\mathcal{R}_S HZ_1(Spl_q(G))$ and $\mathcal{R}_S HZ_1(Sh_q(G))$ with the $\mathcal{R}_S HZ_1(G)$ of base graph $G$. Our study explores the impact of graph operations on spectral features, providing structural insights and analytical connections. The findings enhance our understanding of spectral graph theory and have potential implications in molecular computing and complicated network research, where graph operations are commonly used.

**Keywords:** First Hyper Zagreb matrix, Shadow graph, Splitting graph, First Hyper Zagreb Eigenvalues, First Hyper Zagreb Spectral Radius.

## 1 Introduction

Any regular graph $G$ without isolated vertices will be taken into account here. Randic established the Randic index $\sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}}$, also known as the product-connectivity index, in 1975. A topological index is a numerical invariant that

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characterizes the features of a molecular graph. Randic index has been effectively associated with the chemical and physical properties of organic compounds and has become one of the most important molecular descriptors. Molecular descriptors are invariants of the graph associated with molecular structures [23, 10]. There are various classes of topological indices based on degree, distance, and eigenvalues. The first molecular descriptor is a distance-based invariant known as the Wiener index. Since then different topological indices have been introduced over the years. Topological indices have remarkable uses in chemistry, in the analysis of complex networks, proteins, and bio-informatics. These topics encompass a wide range of biological disciplines, including toxicology, microbiology, virology, and cancer research.

The fundamental reason for the popularity of topological indices is their high flexibility in solving various seemingly unrelated issues in all of these areas in a short period of time. As a result, various intriguing software and theoretical methods for dealing with structure-function information and data mining in this sector have recently been developed. Zagreb indices were initially introduced in [15]. Here, we focus specifically on two topological indices for molecular branching that are seen in both QSAR and QSPR activities. Raza et al. explored key closed relations for the first Zagreb connection index in [20]. The spectral radius reveals information about a graph's connectedness and structure. A larger spectral radius denotes a more complex and interconnected graph, whereas a smaller spectral radius indicates a less interconnected graph. Finding the eigenvalues of an associated matrix, which may be done using many mathematical techniques and software tools, is often required for computing the spectral radius of a graph. It is a useful tool for network analysis and the study of dynamical systems on networks. In 1977 Cvetkovic and Gutman introduced this notion as a measure of branching [9]. Further information was explored in a number of papers, including [9]-[11].

The first and second Zagreb indices are indicators of the molecular carbon-atom skeleton's branching, as seen in [14]. Comprehensive information on the first and second Zagreb indices can be found in [13]. Here, any regular graph will be considered. In the past few years, this field has seen a lot of activity. Recent years have seen the introduction of several spectral radii versions with a range of uses in various fields. One such spectral radius is the First Hyper Zagreb spectral radius, which was developed in consideration of the successful uses of the First Hyper Zagreb indices. How do the spectral radius of the base graph $G$ and the graph created from $G$ relate to one another? This is a logical question. In order to respond to this inquiry, $Spl_q(G)$ and $Sh_q(G)$ have been taken into account and it was found that the First Hyper Zagreb spectral radius of the newly built graph were exactly

recovered from First Hyper Zagreb spectral radius of the base graph.

Bilal et al. discussed the change in ISI energy after one edge deletion [3]. Bilal et al. investigated maximum and minimum degree spectral radii [27], $ABC$ energies and $ABC$ spectral radii [1], $ISI$ energies and $ISI$ spectral radii [6] and Randic and reciprocal Randic spectral radii and energies [2] of the splitting and shadow graphs. Bilal et al. recently answered similar questions related to $SDD$ spectral radii [5], and Modified Sombor spectral invariants [4]. Mobeen et al. investigated Albertson spectral radii and Albertson energies of splitting and shadow graphs [17]. M. R. Oboudi discovered that among all trees, the paths have the greatest and the stars have the smallest reverse Wiener spectral radius [18]. B. A. Rather et al. in [19] determined the distance Laplacian energy of the sun and partial sun graphs. H. A. Ganie et al. determined Signless Laplacian and Laplacian Characteristic Polynomials of Digraph [12]. Y. Shang developed the scaling constant through an iterative process [21]. Information and sources about spectral radii can be found in [7, 8, 26]. The Laplacian spectral radius has a broad applicability in molecular stability and chemical energy levels [25].

Graph operations, particularly splitting and shadow graph operations, enable the development of bigger networks that have explicit spectral links to basic graphs. These techniques enable controlled scaling for spectral features of chemical and network modelling. Walk-based spectral approaches provide actionable insights for community recognition, influencer identification, and spread modelling, which are useful in social network analysis and epidemiology. The First Hyper Zagreb index is an extension of the traditional Zagreb indices. Authors in [24] compiles the first hyper Zagreb index of various graphs, including wheel, complete, and friendship graphs. The authors of [22] computed the first hyper Zagreb index, which includes several graph operations such as Cartesian product and join. Entries of First Hyper Zagreb matrix $[HZ_1(G)] = c_{ij}$ are given below

$$c_{ij} = \left\{ \begin{array}{cc} (d_i + d_j)^2, & \textit{when } q_i \textit{ is adjacent to } q_j, \\ 0, & otherwise. \end{array} \right\}$$

Where $d_i$ denotes the degree of vertex $i$. Eigenvalues of First Hyper Zagreb matrix are $\varrho_1, \varrho_2, ..., \varrho_n$.

First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(G)$ is defined as

$$\mathcal{R}_S HZ_1(G) = \max_{i=1}^{n} \left|\varrho_i\right|.$$

To generate the $q$-splitting graph $Spl_q(G)$, we add $q$ new vertices for each vertex $v$ in the base graph $G$. These new vertices are connected to $v's$ neighbours, but not to $v$ itself. To generate the $q$-shadow graph $Sh_q(G)$, we generate $q$ copies of the

original graph $G$. If vertices are connected in the original graph, we connect them across the copies as well. For further clarification see [6].

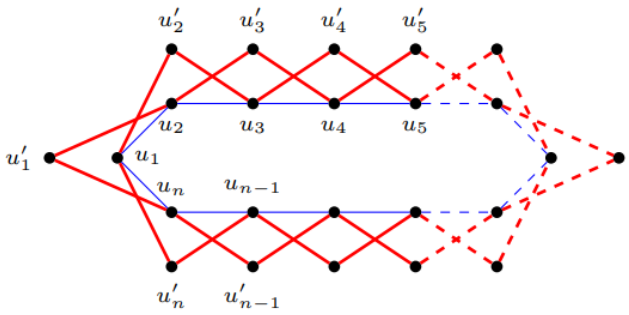


Figure 1: $Spl_1(C_n)$.

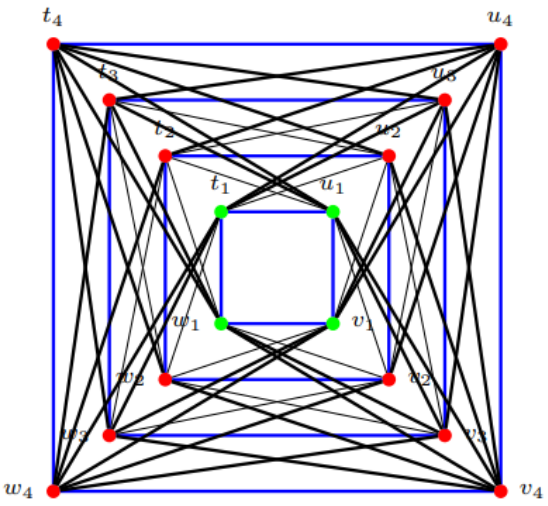


Figure 2: $Sh_4(C_4)$.

Tensor or Kronecker product of $L \epsilon R^{p\times q}$ and $M \epsilon R^{s\times t}$ is

$$L \bigotimes M = \begin{pmatrix} a_{11}M & . & . & .a_{1q}M \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{p1}M & . & . & .a_{pq}M \end{pmatrix}.$$

**Proposition 1.1** ([16]). *Let eigenvalue of $L$ is $\iota$ and eigenvalue of $M$ is $\kappa$. Then $\iota\kappa$ is an eigenvalue of $L \bigotimes M$.*

Proposition 1.1 will be regularly utilized to prove the key results.

## 2 First Hyper Zagreb Spectral Radii of $q$-Splitting Graph of $G$

**Theorem 1.** *$\mathcal{R}_S HZ_1(Spl_q(G))$ and $\mathcal{R}_S HZ_1(G)$ are related as follows:*

$$\mathcal{R}_S HZ_1(Spl_q(G))$$

$$= \left( \frac{2q^2 + 4q + 2 + \sqrt{q^5 + 12q^4 + 40q^3 + 56q^2 + 32q + 4}}{4} \right) (\mathcal{R}_S HZ_1(G)) .$$

*Proof.*

$$HZ_1\left(Spl_q(G)\right) = \begin{bmatrix} \aleph_1 & \aleph_2 & \dots & \aleph_2 \\ \aleph_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \aleph_2 & 0 & \cdots & 0 \end{bmatrix}.$$

$\aleph_1 = (q+1)^2 HZ_1$ and $\aleph_2 = \left(\frac{q+2}{2}\right)^2 HZ_1$.

$$HZ_1\left(Spl_q(G)\right) = \begin{bmatrix} (q+1)^2 HZ_1 & \left(\frac{q+2}{2}\right)^2 HZ_1 & \dots & \left(\frac{q+2}{2}\right)^2 HZ_1 \\ \left(\frac{q+2}{2}\right)^2 HZ_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{q+2}{2}\right)^2 HZ_1 & 0 & \cdots & 0 \end{bmatrix}_{qn+n \times qn+n}$$

$$HZ_1\left(Spl_q(G)\right) = HZ_1 \bigotimes \begin{bmatrix} (q+1)^2 & \left(\frac{q+2}{2}\right)^2 & \dots & \left(\frac{q+2}{2}\right)^2 \\ \left(\frac{q+2}{2}\right)^2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{q+2}{2}\right)^2 & 0 & \cdots & 0 \end{bmatrix}.$$

Let $[F] = f_{ij}$ having entries

$$f_{ij} = \begin{bmatrix} (q+1)^2 & \left(\frac{q+2}{2}\right)^2 & \dots & \left(\frac{q+2}{2}\right)^2 \\ \left(\frac{q+2}{2}\right)^2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{q+2}{2}\right)^2 & 0 & \cdots & 0 \end{bmatrix}.$$

We are particularly interested in $\mathcal{R}_S HZ_1(Spl_q(G))$. It is necessary to compute the eigenvalues of $F$. The eigenvalues of $[F]$ are currently being calculated. $[F]$ has just two non-zero eigenvalue due to its rank.

$$\ell_1 + \ell_2 = tr(F) = (q+1)^2. \tag{2.1}$$

Consider $[F^2] = g_{ij}$ having entries

$$g_{ij} = \begin{bmatrix} (q+1)^4 + q(\frac{q+2}{2})^4 & (q+1)^2\left(\frac{q+2}{2}\right)^2 & \dots & (q+1)^2\left(\frac{q+2}{2}\right)^2 \\ (q+1)^2\left(\frac{q+2}{2}\right)^2 & \left(\frac{q+2}{2}\right)^4 & \dots & \left(\frac{q+2}{2}\right)^4 \\ \vdots & \vdots & \ddots & \vdots \\ (q+1)^2\left(\frac{q+2}{2}\right)^2 & \left(\frac{q+2}{2}\right)^4 & \dots & \left(\frac{q+2}{2}\right)^4 \end{bmatrix}_{q+1\times q+1}$$

Then

$$\ell_1^2 + \ell_2^2 = tr(F^2) = (q+1)^4 + 2q\left(\frac{q+2}{2}\right)^4. \tag{2.2}$$

When equations $(2.1)$ and $(2.2)$ are solved, the following results are obtained

$$\ell_1 = \frac{2q^2+4q+2+\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4}, \tag{2.3}$$

and

$$\ell_2 = \frac{2q^2+4q+2-\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4}. \tag{2.4}$$

$$Ch(F) = \ell^{q-1}(\ell - (\tfrac{2q^2+4q+2+\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4}))$$
$$(\ell - (\tfrac{2q^2+4q+2-\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4})) = 0.$$

So, $specF =$

$$\begin{pmatrix} 0 & \frac{2q^2+4q+2+\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4} & \frac{2q^2+4q+2-\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4} \\ q-1 & 1 & 1 \end{pmatrix}. \tag{2.5}$$

Since $HZ_1(Spl_q(G)) = Z_1 \bigotimes F$. If the eigenvalues of $HZ_1$ are $\varrho_1, \varrho_2, \varrho_3, ...\varrho_n$, then utilizing Proposition 1.1

$$\begin{aligned} &\mathcal{R}_S HZ_1(Spl_q(G) \\ &= max_{i=1}^{n} |(specF))\, \varrho_i| \\ &= max_{i=1}^{n} |\varrho_i|[max\, |specF|] \\ &= \mathcal{R}_S HZ_1(G) \left( \frac{2q^2+4q+2+\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4} \right). \end{aligned}$$

□

**Proposition 2.1.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Spl_q(C_p))$ is*

$$\mathcal{R}_S HZ_1(Spl_q(C_p)) = 8(2q^2+4q+2+\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}).$$

*Proof.* $HZ_1\,(C_p) =$

$$\begin{bmatrix} 0 & (d_1+d_2)^2 & 0 & 0 & \cdots & 0 & (d_1+d_p)^2 \\ (d_2+d_1)^2 & 0 & (d_2+d_3)^2 & 0 & \cdots & 0 & 0 \\ 0 & (d_3+d_2)^2 & 0 & (d_3+d_4)^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ (d_p+d_1)^2 & 0 & 0 & 0 & \cdots & (d_p+d_{p-1})^2 & 0 \end{bmatrix}.$$

$$HZ_1\,(C_p) = \begin{bmatrix} 0 & 16 & 0 & 0 & \cdots & 0 & 16 \\ 16 & 0 & 16 & 0 & \cdots & 0 & 0 \\ 0 & 16 & 0 & 16 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 16 & 0 & 0 & 0 & \cdots & 16 & 0 \end{bmatrix}.$$

The First Zagreb spectrum of $C_p$ is easily observed to be as follows

$$specHZ_1\,(C_p) = 32cos\left(\frac{2\pi t}{p}\right)$$

where $t = 0, 1, \cdots, p-1$.

We arrive at $\mathcal{R}_S HZ_1(C_p)$= 32. Theorem 1 can be employed to obtain the desired outcome. □

**Example 2.1.** *It is easy to see that the First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Spl_1(C_4))$ of $C_4$ is $\mathcal{R}_S HZ_1(Spl_1(C_4)) = 64 + 8\sqrt{145}$*

$$HZ_1(C_4) = \begin{bmatrix} 0 & 16 & 0 & 16 \\ 16 & 0 & 16 & 0 \\ 0 & 16 & 0 & 16 \\ 16 & 0 & 16 & 0 \end{bmatrix}.$$

*and*

$$HZ_1(Spl_1(C_4)) = \begin{bmatrix} 0 & 64 & 0 & 64 & 0 & 36 & 0 & 36 \\ 64 & 0 & 64 & 0 & 36 & 0 & 36 & 0 \\ 0 & 64 & 0 & 64 & 0 & 36 & 0 & 36 \\ 64 & 0 & 64 & 0 & 36 & 0 & 36 & 0 \\ 0 & 36 & 0 & 36 & 0 & 0 & 0 & 0 \\ 36 & 0 & 36 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 36 & 0 & 0 & 0 & 0 \\ 36 & 0 & 36 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Spectrum of $HZ_1(Spl_1(C_4))$ is easily observed to be as follows*

$$specHZ_1(Spl_1(C_4)) = \begin{pmatrix} 0 & 64+8\sqrt{145} & 64-8\sqrt{145} & -64+8\sqrt{145} & -64-8\sqrt{145} \\ 4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

*Utilizing $specHZ_1(Spl_1(C_4))$ we arrive at $\mathcal{R}_S HZ_1(Spl_1(C_4)) = 64 + 8\sqrt{145}$. We now confirm the outcome from proposition 2.1 as follows*

$$\begin{aligned} \mathcal{R}_S HZ_1(Spl_1(C_4)) &= 8(2+4+2+\sqrt{1+12+40+56+32+4}) \\ &= 8(8+\sqrt{145}) = 64+8\sqrt{145}. \end{aligned}$$

**Proposition 2.2.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Spl_q(K_p))$ of $K_p$ is*

$$\mathcal{R}_S HZ_1(Spl_q(K_p)) = \left(\frac{2q^2+4q+2+\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4}\right) \left(4p^3-12p^2+12p-4\right).$$

*Proof.*

$$HZ_1(K_p) = \begin{bmatrix} 0 & (d_1+d_2)^2 & (d_1+d_3)^2 & \dots & (d_1+d_p)^2 \\ (d_2+d_1)^2 & 0 & (d_2+d_3)^2 & \dots & (d_2+d_p)^2 \\ (d_3+d_1)^2 & (d_3+d_2)^2 & 0 & \dots & (d_3+d_p)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (d_p+d_1)^2 & (d_p+d_2)^2 & (d_1+d_3)^2 & \dots & 0 \end{bmatrix}.$$

$$HZ_1(K_p) = \begin{bmatrix} 0 & (2p-2)^2 & (2p-2)^2 & \dots & (2p-2)^2 \\ (2p-2)^2 & 0 & (2p-2)^2 & \dots & (2p-2)^2 \\ (2p-2)^2 & (2p-2)^2 & 0 & \dots & (2p-2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (2p-2)^2 & (2p-2)^2 & (2p-2)^2 & \dots & 0 \end{bmatrix}.$$

First Hyper Zagreb spectrum of $K_p$ is easily observed to be as follows

$$specHZ_1(K_p) = \begin{pmatrix} (p-1)(4p^2-8p+4) & -4p^2+8p-4 \\ 1 & p-1 \end{pmatrix}.$$

We arrive at $\mathcal{R}_S HZ_1(K_p) = 4p^3 - 12p^2 + 12p - 4$. Theorem 1 can be employed to obtain the desired outcome. □

**Example 2.2.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Spl_1(K_4))$ of $K_4$ is*

$$\mathcal{R}_S HZ_1(Spl_1(K_4)) = 18 + 9\sqrt{13}$$

$$HZ_1(K_4) = \begin{bmatrix} 0 & 36 & 36 & 36 \\ 36 & 0 & 36 & 36 \\ 36 & 36 & 0 & 36 \\ 36 & 36 & 36 & 0 \end{bmatrix}.$$

$$HZ_1(Spl_1(K_4)) = \begin{bmatrix} 0 & 144 & 144 & 144 & 0 & 81 & 81 & 81 \\ 144 & 0 & 144 & 144 & 81 & 0 & 81 & 81 \\ 144 & 144 & 0 & 144 & 81 & 81 & 0 & 81 \\ 144 & 144 & 144 & 0 & 81 & 81 & 81 & 0 \\ 0 & 81 & 81 & 81 & 0 & 0 & 0 & 0 \\ 81 & 0 & 81 & 81 & 0 & 0 & 0 & 0 \\ 81 & 81 & 0 & 81 & 0 & 0 & 0 & 0 \\ 81 & 81 & 81 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Spectrum of $HZ_1\left(Spl_1(K_4)\right)$ is easily observed to be as follows*

$$specHZ_1\left(Spl_1(K_4)\right) = \begin{pmatrix} 216+27\sqrt{145} & 216-27\sqrt{145} & -72+9\sqrt{145} & -72-9\sqrt{145} \\ 1 & 1 & 3 & 3 \end{pmatrix}.$$

*Utilizing $specHZ_1\left(Spl_1(K_4)\right)$ we arrive at $\mathcal{R}_S HZ_1(Spl_1(K_4)) = 216+27\sqrt{145}$. We now confirm the outcome from Corollary 2.2 as follows*

$$\begin{aligned}\mathcal{R}_S HZ_1(Spl_1(K_4)) &= \left(\tfrac{2+4+2+\sqrt{1+12+40+56+32+4}}{4}\right)(256-192+48-4)\\ &= \left(\tfrac{8+\sqrt{145}}{4}\right)(108) = 216+27\sqrt{145}.\end{aligned}$$

**Proposition 2.3.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Spl_q(K_{p,p}))$ of $K_{p,p}$ is*

$$\mathcal{R}_S HZ_1(Spl_q(K_{p,p})) = \left(p^3\right)\left(2q^2+4q+2+\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}\right).$$

*Proof.* $HZ_1\left(K_{p,p}\right)$

$$= \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & (d_1+d_{1'})^2 & (d_1+d_{2'})^2 & \dots & (d_1+d_{p'})^2 \\ 0 & 0 & \dots & 0 & (d_2+d_{1'})^2 & (d_2+d_{2'})^2 & \dots & (d_2+d_{p'})^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & (d_p+d_{1'})^2 & (d_p+d_{2'})^2 & \dots & (d_p+d_{p'})^2 \\ \hline (d_{1'}+d_1)^2 & (d_{1'}+d_2)^2 & \dots & (d_{1'}+d_p)^2 & 0 & 0 & \dots & 0 \\ (d_{2'}+d_1)^2 & (d_{2'}+d_2)^2 & \dots & (d_{2'}+d_p)^2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (d_{p'}+d_1)^2 & (d_{p'}+d_2)^2 & \dots & (d_{p'}+d_p)^2 & 0 & 0 & \dots & 0 \end{array}\right].$$

$$HZ_1\left(K_{p,p}\right) = \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & 4p^2 & 4p^2 & \dots & 4p^2 \\ 0 & 0 & \dots & 0 & 4p^2 & 4p^2 & \dots & 4p^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 4p^2 & 4p^2 & \dots & 4p^2 \\ \hline 4p^2 & 4p^2 & \dots & 4p^2 & 0 & 0 & \dots & 0 \\ 4p^2 & 4p^2 & \dots & 4p^2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 4p^2 & 4p^2 & \dots & 4p^2 & 0 & 0 & \dots & 0 \end{array}\right].$$

First Hyper Zagreb spectrum of $K_{p,p}$ is easily observed to be as follows

$$specHZ_1(K_{p,p}) = \begin{pmatrix} 4p^3 & -4p^3 & 0 \\ 1 & 1 & 2p-2 \end{pmatrix}.$$

We arrive at $\mathcal{R}_S HZ_1(K_{p,p}) = 4p^3$. Theorem 1 can be employed to obtain the desired outcome. □

**Proposition 2.4.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Spl_q(\Upsilon))$ of crown is*

$$\mathcal{R}_S HZ_1(Spl_q(\Upsilon)) = \left(\frac{2q^2+4q+2+\sqrt{q^5+12q^4+40q^3+56q^2+32q+4}}{4}\right)\left(4p^3-12p^2+12p-4\right).$$

*Proof.* $HZ_1(\Upsilon)$

$$= \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & 0 & (d_1+d_{2'})^2 & \dots & (d_1+d_{p'})^2 \\ 0 & 0 & \dots & 0 & (d_2+d_{1'})^2 & 0 & \dots & (d_2+d_{p'})^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & (d_p+d_{1'})^2 & (d_p+d_{2'})^2 & \dots & 0 \\ \hline 0 & (d_{1'}+d_2)^2 & \dots & (d_{1'}+d_p)^2 & 0 & 0 & \dots & 0 \\ (d_{2'}+d_1)^2 & 0 & \dots & (d_{2'}+d_p)^2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (d_{p'}+d_1)^2 & (d_{p'}+d_2)^2 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}\right].$$

$$HZ_1(\Upsilon) = \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & 0 & (2p-2)^2 & \dots & (2p-2)^2 \\ 0 & 0 & \dots & 0 & (2p-2)^2 & 0 & \dots & (2p-2)^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & (2p-2)^2 & (2p-2)^2 & \dots & 0 \\ \hline 0 & (2p-2)^2 & \dots & (2p-2)^2 & 0 & 0 & \dots & 0 \\ (2p-2)^2 & 0 & \dots & (2p-2)^2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (2p-2)^2 & (2p-2)^2 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}\right].$$

First Hyper Zagreb spectrum of $\Upsilon$ is easily observed to be as follows

$$specHZ_1(\Upsilon) = \begin{pmatrix} 4p^2-8p+4 & -4p^2+8p-4 & p-1(4p^2-8p+4) & p-1(-4p^2+8p-4) \\ p-1 & p-1 & 1 & 1 \end{pmatrix}.$$

We arrive at $\mathcal{R}_S HZ_1(\Upsilon) = 4p^3-12p^2+12p-4$. Theorem 1 can be employed to obtain the desired outcome. □

# 3 First Hyper Zagreb Spectral Radii of $q$-Shadow Graph of $G$

**Theorem 2.** *The relation between $\mathcal{R}_S HZ_1(Sh_q(G))$ and $\mathcal{R}_S HZ_1(G)$ is*

$$\mathcal{R}_S HZ_1(Sh_q(G)) = q^3 \mathcal{R}_S HZ_1(G).$$

*Proof.*

$$HZ_1\left(Spl_q(G)\right) = \begin{bmatrix} \aleph_5 & \aleph_5 & \dots & \aleph_5 \\ \aleph_5 & \aleph_5 & \dots & \aleph_5 \\ \vdots & \vdots & \ddots & \vdots \\ \aleph_5 & \aleph_5 & \cdots & \aleph_5 \end{bmatrix}.$$

Matrix $\aleph_5$ is given by $\aleph_5 = q^2 HZ_1$. In fact $\aleph_5$ is scalar multiple of the First First Hyper Zagreb matrix.

$$HZ_1\left(Sh_q(G)\right) = HZ_1 \bigotimes \begin{bmatrix} q^2 & q^2 & \dots & q^2 \\ q^2 & q^2 & \dots & q^2 \\ \vdots & \vdots & \ddots & \vdots \\ q^2 & q^2 & \dots & q^2 \end{bmatrix}.$$

Let $[F] = f_{ij}$ having entries

$$f_{ij} = \begin{bmatrix} q^2 & q^2 & \dots & q^2 \\ q^2 & q^2 & \dots & q^2 \\ \vdots & \vdots & \ddots & \vdots \\ q^2 & q^2 & \dots & q^2 \end{bmatrix}_q$$

We are particularly interested in $\mathcal{R}_S HZ_1(Sh_q(G))$, thus we must determine all eigenvalues of $F$. The eigenvalues of $[F]$ are currently being calculated. $[F]$ has just one non-zero eigenvalue due to its rank. Finally, $Ch(F)$ is $Ch(F) = \ell^{q-1}(\ell - q^3) = 0$.
So,

$$specF = \begin{pmatrix} 0 & q^3 \\ q-1 & 1 \end{pmatrix}. \tag{3.1}$$

Since $HZ_1(Sh_q(G)) = HZ_1 \bigotimes F$. If the eigenvalues of $HZ_1$ are $\varrho_1, \varrho_2, \varrho_3, ...\varrho_n$, then utilizing Proposition 1.1

$$\begin{aligned} \mathcal{R}_S HZ_1(Sh_q(G)) &= max_{i=1}^n \left|(specF_{ij}))\,\Lambda\varrho_i\right| \\ &= max_{i=1}^n |\Lambda\varrho_i|[max\left|q^3\right|] \\ &= q^3 \mathcal{R}_S HZ_1(G). \end{aligned}$$

□

**Proposition 3.1.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Sh_q(K_p))$ of $K_p$ is*

$$\mathcal{R}_S HZ_1(Sh_q(K_p)) = 4p^3q^3 - 12p^2q^3 + 12pq^3 - 4q^3.$$

*Proof.* By applying some simple algebra on the spectrum,

$$specHZ_1(K_p) = \begin{pmatrix} (p-1)(4p^2-8p+4) & -4p^2+8p-4 \\ 1 & p-1 \end{pmatrix}.$$

we arrive at $\mathcal{R}_S HZ_1(K_p) = 4p^3 - 12p^2 + 12p - 4$. Theorem 2 can be employed to obtain the desired outcome. □

**Proposition 3.2.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Sh_q(K_{p,p}))$ of $K_{p,p}$ is*

$$\mathcal{R}_S HZ_1(Sh_q(K_{p,p})) = 4p^3q^3.$$

*Proof.* By applying some simple algebra on the spectrum,

$$specHZ_1(K_{p,p}) = \begin{pmatrix} 4p^3 & -4p^3 & 0 \\ 1 & 1 & 2p-2 \end{pmatrix}.$$

We arrive at $\mathcal{R}_S HZ_1(K_{p,p}) = 4p^3$. Theorem 2 can be utilized to obtain the required outcome. □

**Proposition 3.3.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Sh_q(\Upsilon))$ of crown graph is*

$$\mathcal{R}_S HZ_1(Sh_q(\Upsilon)) = 4p^3q^3 - 12p^2q^3 + 12pq^3 - 4q^3.$$

*Proof.* By applying some simple algebra on the spectrum,
$specHZ_1(\Upsilon)$

$$= \begin{pmatrix} 4p^2-8p+4 & -4p^2+8p-4 & p-1(4p^2-8p+4) & p-1(-4p^2+8p-4) \\ p-1 & p-1 & 1 & 1 \end{pmatrix}.$$

We arrive at $\mathcal{R}_S HZ_1(\Upsilon) = 4p^3 - 12p^2 + 12p - 4$. Theorem 2 can be utilized to obtain the required outcome. □

**Proposition 3.4.** *First Hyper Zagreb spectral radius $\mathcal{R}_S HZ_1(Sh_q(C_p))$ of $C_p$ is*

$$\mathcal{R}_S HZ_1(Sh_q(C_p)) = 32q^3.$$

*Proof.* By applying some simple algebra on the spectrum, $specHZ_1(C_p) = 32cos\left(\frac{2\pi t}{p}\right)$ where $t = 0, 1, \cdots, p-1$. We arrive at $\mathcal{R}_S HZ_1(C_p) = 32$. Theorem 2 can be employed to obtain the desired outcome. □

## 4 Conclusions

The spectral radius is a new graph invariant which is used in networking and computer science. First Hyper Zagreb spectral radius is the graph invariant that will be useful for scientists working in molecular chemistry, physics, and computer sciences. The present article has been devoted to establishing new closed relations between the First Hyper Zagreb spectral radii of generalized splitting and shadow graphs constructed on a regular, simple, and connected graph. We arrived at the conclusion that the First Hyper Zagreb spectral radius of the newly developed graphs can be exactly recovered from the First Hyper Zagreb spectral radius of base graphs. We derived these specific relations for some families of graphs like crown graph, $C_p$, $K_p$, and $K_{p,p}$ as propositions.

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